Analysis, Control and Safety of Learning and Piecewise Affine Systems

1 Context and objectives

Emerging applications such as autonomous vehicles and human assistance devices have increased the demand for safety verification, energy efficiency, and cost reduction. Moreover, technological advances led to the development of cheaper sensors, actuators, and processing units, these devices enable operation in changing environments and data collection. The stringent requirements and new technologies impact the feedback control strategies adopted in these applications. Indeed, the control architectures and algorithms may be based on optimization strategies [1], incorporate logic [2], learning algorithms [3], and operate on networks. In this scenario, strategies aiming to provide safety, autonomy, environment adaptation, and constraint handling, lead to nonlinear elements in control laws.

Handling state- and input-constraints can result in nonlinear feedback laws, among which we find the class of *piecewise affine (PWA) control systems*, where the state space is partitioned in sets with distinct dynamics. For this class of systems, models can be explicit as the continuous-time model $\dot{x} = A_i x + b_i$, $x \in \Omega_i$, $\cup_i \Omega_i = \mathbb{R}^n$ or implicit, as discussed in [4]. Even in low dimensions, PWA elements can lead to complex behaviour such as chaotic trajectories and limit cycles [5, 6]. Relevant PWA control systems include: 1) Systems containing loops with Neural Networks (NN) with Rectifier Linear Units (ReLU) activation functions [7, 8, 9]; 2) Model Predictive Control (MPC) strategies [10]; and 3) control loops containing sensor and actuator nonlinearities [11]. Furthermore, since feedback systems will increasingly operate as components of large scale and interconnected systems, heterogeneous control strategies that include learning algorithms, optimisation-based control and constraint handling require scalable methods. Finding common features, such as their PWA description, will help develop unified analysis tools for the interconnected systems. As a result, more sophisticated analysis methods are needed to study these control strategies and handle constraints.

For learning systems, the ReLU function, or ramp function, has been successfully applied to deep learning algorithms in feedback loops capable of handling uncertainty, such as ground effects in drones or environment evaluation for robotics [12, 3]. Moreover, the current trend on using data and learning algorithms will further promote NN's as approximations of nonlinear elements in dynamical systems that are difficult to model, that vary in time, or that are subject to disturbances. A rigorous analysis of the impact of these nonlinear elements in closed-loop is still lacking, even for linear systems subject to state- and input-constraints.

The project's main ***objective*** is to develop methods for a rigorous analysis of the stability and safety of systems containing piecewise affine nonlinear elements. In particular, we will study systems having Neural Networks in closed-loop, PWA systems obtained from MPC strategies for constrained linear systems, or systems with PWA functions introduced by sensors and actuators limitations. For dynamical systems with NN, certificates for stability and safety will be computed to guarantee safe learning and operation.

To introduce a rigorous and non-conservative analysis of dynamical systems with piecewise affine functions, we have recently proposed [13] an **implicit representation** of the PWA function as

$$f(x) = Ax + B\phi(y) \tag{1a}$$

$$y = Cx + D\phi(y) + e \tag{1b}$$

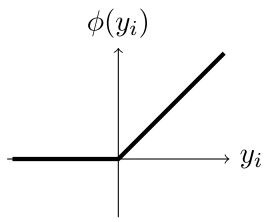

Figure 1: Function elements of $\phi(y)$.

with ϕ the vector of ramp functions (see Figure 1). Crucially, element-wise, the nonlinear vector function ϕ satisfies the inequalities

$$\phi(y) \geq 0,\ (\phi(y) - y) \geq 0,\ \phi(y)(\phi(y) - y) = 0, \tag{2}$$

which are linear complementarity properties obtained with the Karush-Kuhn-Tucker optimality conditions [14]. Using KKT conditions for Quadratic Programs as in [10], we can cast MPC laws as (1b), namely, in terms of ramp functions. We have recently shown [15] that the implicit relation in (1b) can generate a set-valued step function, thus enabling the representation of *discontinuous* PWA functions.

The ability to describe discontinuous functions with (1) is key to study control laws described by automata [16] or actuators able to deliver a finite set of inputs or composed by switches and relays [17, 18]. In this case, state feedback laws can be given in terms of a partition of the state-space where each input level is active.

The main existing **methods** to analyse stability, robustness, and safety for piecewise affine systems are based on Lyapunov analysis. A Lyapunov function or a storage function is then the certificate for the studied properties. The current analysis methods using explicit representations are limited to a few state variables or sets in the partition. The framework we propose will circumvent these difficulties as piecewise quadratic Lyapunov functions are naturally inherited by encapsulating the PWA terms in generalised quadratic forms [13, 15]. The main advantage of such a structure is a convex optimisation formulation for the studied problems thanks to the relations (2) as in [13, 19].

Even though the analysis of activation functions in NN as sector nonlinearities has already been proposed [20], *the exact description of the ramp function using* (2) *and the use of Piecewise Quadratic functions together with scalable computational methods gives new perspectives for the analysis of learning systems* [19]. Following a similar direction, the relations (2) have recently been used in the reachability analysis of static maps for sets of inputs [7]. The effects of ReLU elements in a **dynamical system** with state and actuator constraints have yet to be rigorously studied. In the case of feedforward ReLU NN with an arbitrary number of layers, the NN is directly in the form (1) with (1b) presenting a strict block-triangular D matrix, which implies that (1b) leads to an *explicit* PWA representation expressed in terms of ramp functions. An important question is whether it is possible to convert between this explicit and an implicit, but more compact, expression. A strict block-triangular D simplifies the solution to (1b); however, a compact expression can be suitable for embedded controls since it can reduce the memory requirements for storage and evaluation.

The above question is related to the search of **NN to represent MPC** control laws [21, 22]. MPC laws have an explicit PWA representation, and their evaluation requires the solution of a point location problem that may not be scalable [23, 24, 25]. Alternatives to speed up the function evaluation are sought. Searching for feedforward ReLU NNs to approximate MPC (thus yielding a structured matrix D in (1b)) provides PWA functions that only apply simple operations, leading to faster function evaluation. Therefore, establishing a method to convert between an implicit and compact representation of a PWA function (1) to an explicit but structured and larger representation of the same function will avoid function approximation schemes [21, 27].

In the context of a constrained set of operation, only **local properties** are required. Thus the mathematical tools have to be adapted to treat the regional analysis. For instance, the inequalities (2) should be modified to hold only locally, in a region around the origin. When global results may not be obtained, assessing local properties, such as estimates of stability regions and local gains, will be performed similarly to the regional analysis of saturating systems [11, 19]. The regional analysis is of paramount importance for nonlinear systems, particularly for time-varying systems, like those described by online learning algorithms.

It is essential to evaluate the real-time implementation **robustness** to guarantee the efficiency of the PWA function evaluation. If the implicit equation (1b) has to be solved in a real-time scenario to generate a control input, it might not be computed exactly, and the error in the solution acts as a disturbance. We will rely on the Input-to-State stability formalism to evaluate the impact of these numerical errors in the system trajectories. Another important robustness property is related to uncertainty in the partition of PWA functions, challenging to cope with existing stability analysis methods. The use of the implicit equation (1b) to describe the partition helps parameterize its uncertainty as parametric uncertainties in the matrices of (1b). Finally, we will introduce state-dependent disturbances as envelopes of piecewise functions, a case of practical interest as in autonomous vehicle applications, where disturbances may vary according to the position (due to sensor precision depending on obstacle position).

The computation of invariant sets for robustness will also be used for **safety assessment**. The use the quadratic relations (2) to verify piecewise quadratic inequalities, will also be used to compute invariant sets. These sets can characterise the nominal operation of quantised systems and describe reachable sets under disturbances. We can conclude that a trajectory is *safe* if the invariant sets do not intersect *unsafe* regions of the state space. A similar goal has been recently pursued in [28]. It is motivated by the fact that PWA systems stemming from NN must be certified since NN approximation properties may fail [29]. Such mismatch on a dynamical system may violate safety constraints. Also, in [19] simple quadratic Lyapunov functions were studied for linear systems in feedback with NN controllers using Integral Quadratic Constraints (IQC).

To illustrate the theoretical and numerical methods yielding stability and safety certificates and robustness evaluation in terms of reachable sets, we will focus on the particular application of **autonomous vehicles**. This application is the most suitable for the studied problems since MPC [31], thanks to its constraint handling capabilities, and reinforcement learning have been extensively used for path planning, and safety manoeuvres [32, 33, 34]. However, in most cases, only high-level kinematics models or abstraction, ignoring dynamics, have been studied [35, 36] and safety as well as the stability analysis including sensor and actuator constraints are not carried out. More recently, an NN-based feedback controller has been applied to autonomous racing vehicles; however, it cannot guarantee safety [37].

1.1 Main objectives

Objective 1 - Stability and Safety for PWA and learning Systems: Closed loop modelling of systems with ReLU NN, MPC schemes and nonlinear actuators as PWA systems; Stability analysis with Lyapunov functions and Safety verification with Barrier functions for PWA systems. Application to NN with ReLU elements: Definition of Regions in the parameter space of the NN for safe learning; **Objective 2** - Robustness for PWA functions and MPC: Reachability analysis in the discrete, continuous and Hybrid contexts; ISS analysis for PWA control errors, including MPC; Computation of invariant sets as methods for safety analysis; State-dependent uncertainty handling by PWA approximation; **Objective 3** - Real-time implementation of PWA control laws: Strategies to implement of PWA functions under the form of (1b); Relations between minimal and explicit representations; Comparison with current solution for the implicit absolute value equation. **Objective 4** - Software development: Toolbox for PWA system analysis. Toolbox functionalities will include model conversion, reduction of PWA functions. Functions for regional and global stability, reachability, and safety assessment; Development, release and maintenance. Dissemination: Further to the scientific reports, the deliverables of the project are 1) scientific publications in the top journals of automatic control and applied mathematics; book chapters in specialised volumes; 2) release software as detailed in WP 4. 3) papers in top conferences and workshops of the field

2 Supervision

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